

# PRINCIPAL BUNDLES OF REDUCTIVE GROUPS OVER AFFINE SCHEMES

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ABSTRACT. Let  $R$  be a semi-local regular domain containing an infinite perfect field  $k$ , and let  $K$  be the field of fractions of  $R$ . Let  $G$  be a reductive semi-simple simply connected  $R$ -group scheme such that each of its  $R$ -indecomposable factors is isotropic. We prove that for any Noetherian affine scheme  $\mathcal{A} = \operatorname{Spec} A$  over  $k$ , the kernel of the map

$$H_{\text{ét}}^1(\mathcal{A} \times_{\operatorname{Spec} k} \operatorname{Spec} R, G) \rightarrow H_{\text{ét}}^1(\mathcal{A} \times_{\operatorname{Spec} k} \operatorname{Spec} K, G)$$

induced by the inclusion of  $R$  into  $K$ , is trivial. If  $R$  is the semi-local ring of several points on a  $k$ -smooth scheme, then it suffices to require that  $k$  is infinite and keep the same assumption concerning  $G$ . The results extend the Serre—Grothendieck conjecture for such  $R$  and  $G$ , proved in [PaStV].

## 1. INTRODUCTION

Recall that an  $R$ -group scheme  $G$  is called *reductive* (respectively, *semi-simple*; respectively, *simple*), if it is affine and smooth as an  $R$ -scheme and if, moreover, for each ring homomorphism  $s : R \rightarrow \Omega(s)$  to an algebraically closed field  $\Omega(s)$ , its scalar extension  $G_{\Omega(s)}$  is a reductive (respectively, semi-simple; respectively, simple) algebraic group over  $\Omega(s)$ . This notion of a simple  $R$ -group scheme coincides with the notion of a simple semi-simple  $R$ -group scheme of [SGA3, Exp. XIX, Def. 2.7 and Exp. XXIV, 5.3].

Such an  $R$ -group scheme  $G$  is called *simply-connected* (respectively, *adjoint*), if for any homomorphism  $s : R \rightarrow \Omega(s)$  of  $R$  to an algebraically closed field  $\Omega(s)$ , the group  $G_{\Omega(s)}$  is a simply-connected (respectively, adjoint)  $\Omega(s)$ -group scheme (see [SGA3, Exp. XXII, Def. 4.3.3]). A simple group scheme  $G$  is called *isotropic*, if it contains a split torus  $G_{m,R}$ .

We prove the following theorem, which is an extension of the results on the Serre—Grothendieck conjecture obtained in [PaStV].

**Theorem 1.1.** *Let  $R$  be a regular semi-local domain containing a infinite perfect field  $k$ . Let  $K$  be the field of fractions of  $R$ . Let  $G$  be an isotropic simple simply connected  $R$ -group scheme.*

*For any Noetherian affine scheme  $\mathcal{A} = \operatorname{Spec} A$  over  $k$ , the map*

$$H_{\text{ét}}^1(\mathcal{A} \times_{\operatorname{Spec} k} \operatorname{Spec} R, G) \rightarrow H_{\text{ét}}^1(\mathcal{A} \times_{\operatorname{Spec} k} \operatorname{Spec} K, G)$$

*induced by the inclusion of  $R$  into  $K$ , has trivial kernel.*

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This theorem is deduced, via a theorem of D. Popescu, from its following “geometric” version.

**Theorem 1.2.** *Let  $R$  be a semi-local ring of several points on a  $k$ -smooth scheme over an infinite field  $k$ . Let  $K$  be the field of fractions of  $R$ . Let  $G$  be an isotropic simple simply connected  $R$ -group scheme.*

*For any Noetherian affine scheme  $\mathcal{A} = \operatorname{Spec} A$  over  $k$ , the map*

$$H_{\text{ét}}^1(\mathcal{A} \times_{\operatorname{Spec} k} \operatorname{Spec} R, G) \rightarrow H_{\text{ét}}^1(\mathcal{A} \times_{\operatorname{Spec} k} \operatorname{Spec} K, G)$$

*induced by the inclusion of  $R$  into  $K$ , has trivial kernel.*

The proof of this theorem is given in Section 3. It uses, on one hand, the constructions and results of [PaStV]. On the other hand, it requires the following extension of Quillen’s local-global principle for projective modules [Q, Theorem 1], due to L.-F. Moser (previously announced without proof by Raghunathan [R1, Theorem 2], and hinted in [BCW]).

**Theorem.** [Mo, Korollar 3.5.2] *Let  $A$  be a Noetherian commutative ring,  $G$  a group scheme over  $A$  admitting a closed embedding  $G \rightarrow \operatorname{GL}_{n,A}$  for some  $n \geq 1$ . Let  $E$  be a  $G$ -torsor over  $\mathbb{A}_A^1$ , such that  $E$  is trivial on  $\mathbb{A}_{U_i}^1$  for all elements  $U_i$  of a Zariski covering  $\operatorname{Spec} A = \bigcup U_i$ , and on the zero-section  $\{0\} \times \operatorname{Spec} A$ . Then  $E$  is trivial.*

Using this local-global principle one more time, we obtain the following corollary of Theorem 1.1.

**Corollary 1.3.** *Let  $S$  be a Noetherian ring such that for any maximal ideal  $m$  of  $S$ , the local ring  $S_m$  satisfies the conditions imposed on  $R$  in Theorem 1.1, or in Theorem 1.2. Let  $G$  be a simple simply connected  $S$ -group scheme admitting a closed embedding  $G \rightarrow \operatorname{GL}_{n,S}$  for some  $n \geq 1$ , and such that for any maximal ideal  $m$  of  $S$ , the group  $G_{S_m}$  is isotropic. Let  $K$  be the field of fractions of  $S$ . Then the natural map*

$$H_{\text{ét}}^1(S[t], G) \rightarrow H_{\text{ét}}^1(K(t), G)$$

*has trivial kernel.*

*Proof.* Consider the composition

$$H_{\text{ét}}^1(S[t], G) \rightarrow H_{\text{ét}}^1(K[t], G) \rightarrow H_{\text{ét}}^1(K(t), G).$$

By [?, Prop. 2.2] the map  $H_{\text{ét}}^1(K[t], G) \rightarrow H_{\text{ét}}^1(K(t), G)$  has trivial kernel. It remains to prove that  $H_{\text{ét}}^1(S[t], G) \rightarrow H_{\text{ét}}^1(K[t], G)$  has trivial kernel. By the local-global principle, we can substitute  $S$  by its localization at a maximal ideal, and then apply Theorem 1.1 for  $\mathcal{A} = \mathbb{A}_k^1$ .  $\square$

**Remark 1.** The conditions of Corollary 1.3 on  $S$  are satisfied, in particular, if  $S$  is a (not necessarily semilocal) regular ring containing an infinite perfect field, or if  $\operatorname{Spec} S$  is a smooth affine scheme over an infinite field.

**Corollary 1.4.** *Let  $S, G$  be as in Corollary 1.3. Assume moreover that the field of fractions  $K$  of  $S$  is perfect. Then the map*

$$H_{\text{ét}}^1(S[t], G) \rightarrow H_{\text{ét}}^1(S, G)$$

induced by evaluation at  $t = 0$ , has trivial kernel.

*Proof.* We have a commutative diagram

$$(1) \quad \begin{array}{ccc} H_{\text{ét}}^1(S[t], G) & \xrightarrow{t=0} & H_{\text{ét}}^1(S, G) \\ \downarrow & & \downarrow \\ H_{\text{ét}}^1(K[t], G) & \xrightarrow{t=0} & H_{\text{ét}}^1(K, G). \end{array}$$

Since  $K$  is perfect, the bottom line is an isomorphism by the main result of [RR]. The left vertical line has trivial kernel by Corollary 1.3.  $\square$

**Remark 2.** The conditions of Corollary 1.4 on  $S$  are satisfied, in particular, if  $S$  is a (not necessarily semilocal) regular ring containing  $\mathbb{Q}$ , or if  $\text{Spec } S$  is a smooth affine scheme over a field of characteristic 0.

**Remark 3.** All the above results can be easily extended to the case where  $G$  is not simple but semisimple, and satisfies the following isotropy condition: every semisimple normal subgroup of  $G$  is isotropic. This follows from Faddeev—Shapiro lemma [SGA3, Exp. XXIV Prop. 8.4] (see also [PaStV, Section 12]).

## 2. CONSTRUCTION OF A BUNDLE OVER AN AFFINE LINE

Let  $k, R, K, G$  be as in Theorem 1.2. Let  $\mathcal{A}$  be any scheme over  $k$ . In this section we show that any (étale) principal  $G$ -bundle over  $\mathcal{A} \times_{\text{Spec } k} \text{Spec } R$  which becomes trivial over  $\mathcal{A} \times_{\text{Spec } k} \text{Spec } K$  can be substituted by a principal  $G$ -bundle  $P_t$  over  $\mathbb{A}_R^1 \times \mathcal{A}$  which is trivial over  $(\mathbb{A}_R^1)_f \times \mathcal{A}$ , for some monic polynomial  $f \in R[t]$ , in such a way that the triviality of this new bundle implies the triviality of  $P$ . The argument is an extension of the argument of [PaStV, §6].

For compatibility with [PaStV], in this section we denote  $R$  by  $\mathcal{O}$ . We set

$$Y := \mathcal{A} \times_{\text{Spec } k} \text{Spec } \mathcal{O} = \mathcal{A} \times_{\text{Spec } k} \text{Spec } R$$

for shortness.

Fix a smooth affine  $k$ -scheme  $X$  and a finite family of points  $x_1, x_2, \dots, x_n$  on  $X$ , such that  $\mathcal{O} = \mathcal{O}_{X, \{x_1, x_2, \dots, x_n\}}$ . Set  $U := \text{Spec}(\mathcal{O})$ . Let

$$\text{can} : U \rightarrow X$$

be the canonical map. Further, consider a simple simply-connected  $U$ -group scheme  $G$ .

Let  $P$  be a principal  $G$ -bundle over the scheme  $Y$  which is trivial over  $Y \times_{\text{Spec } \mathcal{O}} \text{Spec } K$ . We may and will assume that for certain  $f \in \mathcal{O}$  the principal  $G$ -bundle  $P$  is trivial over  $Y \times_{\text{Spec } \mathcal{O}} \text{Spec } \mathcal{O}_f$ . Shrinking  $X$  if necessary, we may secure the following properties

- (i) The points  $x_1, x_2, \dots, x_n$  are still in  $X$ .
- (ii) The group scheme  $G$  is defined over  $X$  and it is a simple group scheme. We will often denote this  $X$ -group scheme by  $G_X$  and write  $G_U$  for the original  $G$ .
- (iii) The principal  $G$ -bundle  $P$  is defined over  $Y \times_{\text{Spec } \mathcal{O}} X$  and the function  $f \in \mathcal{O}$  belongs to  $k[X]$ .

(iv) The restriction  $P_f$  of the bundle  $P$  to the open subset  $Y \times_{\text{Spec } \mathcal{O}} X_f$  is trivial and  $f$  vanishes at each  $x_i$ 's.

In particular, now we are given the smooth irreducible affine  $k$ -scheme  $X$ , the finite family of points  $x_1, x_2, \dots, x_n$  on  $X$ , and the non-zero function  $f \in k[X]$  vanishing at each point  $x_i$ . It was shown in [PaStV, Section 5] that, starting with these data, one can construct what is called there a *nice triple* [PaStV, Def. 4.1], of the form  $(q_U : \mathcal{X} \rightarrow U, f, \Delta)$ . This triple fits into a commutative diagram

$$(2) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{q_X} & X \\ q_U \downarrow & \Delta \nearrow & \text{can} \\ U & & \end{array}$$

where

$$(3) \quad q_X \circ \Delta = \text{can}$$

and

$$(4) \quad q_U \circ \Delta = \text{id}_U.$$

Moreover,  $f := q_X^*(f)$ . We did that shrinking  $X$  along the way, but all properties (i) to (iv) were preserved.

In particular, the restriction  $P_f$  of the bundle  $P$  to the open subscheme  $Y \times_{\text{Spec } \mathcal{O}} X_f$  is trivial by Item (iv) above.

Set  $G_{\mathcal{X}} := (q_X)^*(G)$ , and let  $G_{\text{const}}$  be the pull-back of  $G_U$  to  $\mathcal{X}$  via  $q_U$ . By [PaStV, Theorem 4.3] there exists a *morphism of nice triples* [PaStV, Def. 4.2]

$$\theta : (\mathcal{X}', f', \Delta') \rightarrow (\mathcal{X}, f, \Delta)$$

and an isomorphism

$$(5) \quad \Phi : \theta^*(G_{\text{const}}) \rightarrow \theta^*(G_{\mathcal{X}}) =: G_{\mathcal{X}'}$$

of  $\mathcal{X}'$ -group schemes such that  $(\Delta')^*(\Phi) = \text{id}_{G_U}$ .

Set

$$(6) \quad q'_X = q_X \circ \theta : \mathcal{X}' \rightarrow X.$$

Recall that

$$(7) \quad q'_U = q_U \circ \theta : \mathcal{X}' \rightarrow U,$$

since  $\theta$  is a morphism of nice triples.

Consider the pullback  $(q'_X)^*(P)$  of  $P$  from  $Y \times_U X$  to  $Y \times_U \mathcal{X}'$  as a principal  $(q'_U)^*(G_U) = \theta^*(G_{\text{const}})$ -bundle via the isomorphism  $\Phi$ .

Recall that  $P$  is trivial as a  $G$ -bundle over  $Y \times_U X_f$ . Therefore,  $(q'_X)^*(P)$  is trivial as a principal  $G_{\mathcal{X}'}$ -bundle over  $Y \times_U \mathcal{X}'_{\theta^*(f)}$ . Since  $\theta$  is a nice triple morphism one has  $f' = \theta^*(f) \cdot g'$ , and thus the principal  $G_{\mathcal{X}'}$ -bundle  $(q'_X)^*(P)$  is trivial over  $Y \times_U \mathcal{X}'_{f'}$ .

We conclude that  $(q'_X)^*(P)$  is trivial over  $Y \times_U \mathcal{X}'_{f'}$ , when regarded as a principal  $G_U$ -bundle (more precisely,  $(q'_U)^*(G_U)$ -bundle; we omit this base change from the notation) via the isomorphism  $\Phi$ .

By [PaStV, Theorem 4.5] there exists a finite surjective morphism  $\sigma : \mathcal{X}' \rightarrow \mathbb{A}^1 \times U$  of  $U$ -schemes satisfying

- (1)  $\sigma$  is étale along the closed subset  $\{f' = 0\} \cup \Delta'(U)$ .
- (2) For a certain element  $g_{f',\sigma} \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$  and a unitary polynomial  $N(f') \in \mathcal{O}[t] \hookrightarrow \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ , defined by the distinguished  $\sigma$  as in [PaStV, Section 4], one has

$$\sigma^{-1}\left(\sigma(\{f' = 0\})\right) = \{N(f') = 0\} = \{f' = 0\} \sqcup \{g_{f',\sigma} = 0\}.$$

- (3) Denote by  $(\mathcal{X}')^0 \hookrightarrow \mathcal{X}'$  the largest open sub-scheme, where the morphism  $\sigma$  is étale. Write  $g'$  for  $g_{f',\sigma}$  from now on. Then the square

$$(8) \quad \begin{array}{ccc} (\mathcal{X}')^0_{N(f')} = (\mathcal{X}')^0_{f'g'} & \xrightarrow{\text{inc}} & (\mathcal{X}')^0_{g'} \\ \sigma^0_{f'g'} \downarrow & & \downarrow \sigma^0_{g'} \\ (\mathbb{A}^1 \times U)_{N(f')} & \xrightarrow{\text{inc}} & \mathbb{A}^1 \times U \end{array}$$

is an elementary Nisnevich square. (Here  $\sigma^0_{f'g'}$  and  $\sigma^0_{g'}$  stand for the corresponding restrictions of  $\sigma$ .)

- (4) One has  $\Delta'(U) \subset (\mathcal{X}')^0_{g'}$ .

Regarded as a principal  $G_U$ -bundle via the isomorphism  $\Phi$ , the bundle  $(q'_X)^*(P)$  over  $Y \times_U \mathcal{X}'$  becomes trivial over  $Y \times_U \mathcal{X}'_{f'}$ , and a fortiori over  $Y \times_U (\mathcal{X}')^0_{f'g'}$ . Now, gluing the trivial  $G_U$ -bundle over  $Y \times_U (\mathbb{A}^1 \times U)_{N(f')}$  to the bundle  $(q'_X)^*(P)$  along the isomorphism

$$(9) \quad \psi : Y \times_U (\mathcal{X}')^0_{N(f')} \times_U G_U \rightarrow (q'_X)^*(P)|_{Y \times_U (\mathcal{X}')^0_{N(f')}} \times_U G_U$$

of principal  $G_U$ -bundles, we get a principal  $G_U$ -bundle  $P_t$  over  $Y \times_U (\mathbb{A}^1 \times U)$  such that

1. it is trivial over  $Y \times_U (\mathbb{A}^1 \times U)_{N(f')}$ ,
2.  $(\sigma)^*(P_t)$  and  $(q'_X)^*(P)$  are isomorphic as principal  $G_U$ -bundles over  $Y \times_U (\mathcal{X}')^0_{g'}$ . Here  $(q'_X)^*(P)$  is regarded as a principal  $G_U$ -bundle via the  $\mathcal{X}'$ -group scheme isomorphism  $\Phi$  from (5);
3. over  $Y \times_U (\mathcal{X}')^0_{N(f')}$  the two  $G_U$ -bundles are identified via the isomorphism  $\psi$  from (9).

Finally, form the following diagram

$$(10) \quad \begin{array}{ccccc} & & \sigma^0_{g'} = \sigma|_{(\mathcal{X}')^0_{g'}} & & \\ & & \swarrow & & \searrow \\ \mathbb{A}^1_U & \xleftarrow{\quad} & (\mathcal{X}')^0_{g'} & \xrightarrow{q'_X} & X \\ & \searrow \text{pr} & \downarrow q'_U & \nearrow \text{can} & \\ & & U & & \end{array} \quad \Delta'$$

This diagram is well-defined since by Item (4) above the image of the morphism  $\Delta'$  lands in  $(\mathcal{X}')^0_{g'}$ .

**Lemma 2.1.** *The unitary polynomial  $h = N(f') \in \mathcal{O}[t]$ , the principal  $G_U$ -bundle  $P_t$  over  $Y \times_U \mathbb{A}_U^1$ , the diagram (10) and the isomorphism (5) constructed above has the following properties:*

$$(1^*) \ q'_U = \text{pr} \circ \sigma,$$

$$(2^*) \ \sigma \text{ is étale,}$$

$$(3^*) \ q'_U \circ \Delta' = \text{id}_U,$$

$$(4^*) \ q'_X \circ \Delta' = \text{can},$$

$$(5^*) \ \text{the restriction of } P_t \text{ to } Y \times_U (\mathbb{A}_U^1)_h \text{ is a trivial } G_U\text{-bundle,}$$

(6\*)  $(\sigma)^*(P_t)$  and  $(q'_X)^*(P)$  are isomorphic as principal  $G_U$ -bundles over  $Y \times_U (\mathcal{X}')^0_{g'}$ . Here  $(q'_X)^*(P)$  is regarded as a principal  $G_U$ -bundle via the group scheme isomorphism  $\Phi$ .

*Proof.* By the very choice of  $\sigma$  it is an  $U$ -scheme morphism, which proves (1\*). Since  $(\mathcal{X}')^0 \hookrightarrow \mathcal{X}'$  is the largest open sub-scheme where the morphism  $\sigma$  is étale, one gets (2\*). Property (3\*) holds for  $\Delta'$  since  $(\mathcal{X}', f', \Delta')$  is a nice triple and, in particular,  $\Delta'$  is a section of  $q'_U$ . Property (4\*) can be established as follows:

$$q'_X \circ \Delta' = (q_X \circ \theta) \circ \Delta' = q_X \circ \Delta = \text{can}.$$

The first equality here holds by the definition of  $q'_X$ , see (6); the second one holds, since  $\theta$  is a morphism of nice triples; the third one follows from (3). Property (5\*) is just Property 1 in the above construction of  $P_t$ . Property (6\*) is precisely Property 2 in our construction of  $P_t$ .  $\square$

One readily sees that the properties in Lemma 2.1 imply that if the  $G$ -bundle  $P_t$  is trivial on  $Y \times_U \mathbb{A}_U^1$ , then the original bundle  $P$  is trivial on  $Y$ .

Indeed, if  $P_t$  is trivial, then by Property (6\*) in Lemma 2.1 the  $G_U$ -bundle  $(q'_X)^*(P)$  over  $Y \times_U (\mathcal{X}')^0_{g'}$  is trivial as well. Hence, using Property (4\*), we deduce that the bundle  $(\Delta')^*((q'_X)^*(P)) = \text{can}^*(P)$  is a trivial  $(\Delta')^*((q'_X)^*(G)) = \text{can}^*(G)$ -bundle over  $Y \times_U U = Y$ .

### 3. PROOF OF THEOREMS 1.1 AND 1.2

The following easy lemma was essentially proved inside the proof of [PaStV, Theorem 8.6]. Here we provide a more detailed proof in a slightly more general situation.

**Lemma 3.1.** *Let  $R$  be a semilocal ring,  $G$  a simply connected semisimple group scheme over  $R$ . There exists a closed embedding  $G \rightarrow \text{GL}_{n,R}$  for some  $n \geq 1$ .*

*Proof.* We can assume without loss of generality that  $R$  is connected. Let  $U = \text{Spec } R$ . The  $U$ -group scheme  $G$  is given by a 1-cocycle  $\xi \in Z^1(U, \text{Aut}(G_0))$ , where  $G_0$  is the split simply connected simple group scheme over  $U$  of the same type as  $G$ , and  $\text{Aut}(G_0)$  is the automorphism group scheme of  $G_0$ . Recall that  $\text{Aut}(G_0) \cong G_0^{\text{ad}} \rtimes N$ , where  $N$  is the finite group of automorphisms of the Dynkin diagram of  $G_0$ , and  $G_0^{\text{ad}}$  is the adjoint group corresponding to  $G_0$ . Since  $\text{Aut}(G_0) \cong G_0^{\text{ad}} \rtimes N$ , we have an exact sequence of pointed sets

$$\{1\} \rightarrow H^1(U, G_0^{\text{ad}}) \rightarrow H^1(U, G_0^{\text{ad}} \rtimes N) \rightarrow H^1(U, N).$$

Thus there is a finite étale morphism  $\pi : V \rightarrow U$  such that  $G_V := G \times_U V$  is given by a 1-cocycle  $\xi_V \in Z^1(U, G_0^{ad})$ . We can choose  $V$  so that  $V/U$  is moreover a Galois extension.

For each fundamental weight  $\lambda$  of  $G_0$ , there is a central (also called center preserving, see [PeSt]) representation  $\rho_\lambda : G_0 \rightarrow GL_{V_\lambda \otimes_{\mathbb{Z}} U}$ , where  $V_\lambda$  is the Weyl module over  $\mathbb{Z}$  corresponding to  $\lambda$ . This gives a commutative diagram of  $U$ -group morphisms

$$(11) \quad \begin{array}{ccc} G_0 & \xrightarrow{\rho_\lambda} & GL_{V_\lambda \otimes_{\mathbb{Z}} U} \\ \downarrow & & \downarrow \\ G_0^{ad} & \xrightarrow{\bar{\rho}_\lambda} & PGL_{V_\lambda \otimes_{\mathbb{Z}} U} . \end{array}$$

Considering the product of  $\rho_\lambda$ 's with  $\lambda$  running over the set  $\Lambda$  of all fundamental weights, we obtain the following commutative diagram of algebraic  $k$ -group homomorphisms:

$$(12) \quad \begin{array}{ccc} G_0 & \xrightarrow{\rho} & \prod_{\lambda \in \Lambda} GL_{V_\lambda \otimes_{\mathbb{Z}} U} \\ \downarrow & & \downarrow \\ G_0^{ad} & \xrightarrow{\bar{\rho}} & \prod_{\lambda \in \Lambda} PGL_{V_\lambda \otimes_{\mathbb{Z}} U} . \end{array}$$

By the definition of Weyl modules,  $\rho$  is a closed embedding (cf. [PeSt, Lemma 2]).

Twisting the  $V$ -group morphism  $\rho$  with the 1-cocycle  $\xi_V$  we get an  $V$ -group scheme morphism  $\rho_V : G_V \rightarrow \prod_{\lambda \in \Lambda} GL_1(A_\lambda)$ , where the product is a product of group schemes over  $V$ , and each  $A_\lambda$  is an Azumaya algebra over  $V$  obtained from  $\text{End}(V_\lambda \otimes_{\mathbb{Z}} U)$  via the 1-cocycle  $\theta_\lambda = (\bar{\rho}_\lambda)_*(\xi_V) \in Z^1(V, PGL_{V_\lambda \otimes_{\mathbb{Z}} U})$ . Composing  $\rho_V$  with the natural closed embedding  $\prod_{\lambda \in \Lambda} GL_1(A_\lambda) \hookrightarrow GL_{\oplus A_\lambda}$ , we obtain a closed embedding

$$G_V \hookrightarrow GL_{m,V},$$

for a large enough integer  $m$ .

One has

$$\text{Hom}_V(G_V, GL_{m,V}) = \text{Hom}_U(G, R_{V/U}(GL_{m,V})),$$

where  $R_{V/U}$  is the Weil restriction functor. Thus  $\rho_V$  determines an  $U$ -morphism

$$\rho_U : G \hookrightarrow R_{V/U}(GL_{m,V}).$$

Here  $\rho_U$  is a  $U$ -group scheme morphism, and, since  $\rho$  is a closed embedding,  $\rho_U$  is a closed embedding as well (étale descent).

Let  $d$  be the degree of the Galois extension  $V = \text{Spec } S$  over  $U = \text{Spec } R$ . The  $U$ -group scheme  $R_{V/U}(GL_{m,V})$  admits a natural closed embedding into  $GL_{md,U}$ , such that, for any  $R$ -algebra  $X$ , the image of  $g \in R_{V/U}(GL_{m,V})(X) = GL_m(X \otimes_R S)$  is the corresponding element of  $GL_{md}(X)$ , the  $X$ -linear automorphism of  $X^{\oplus md} \cong (X \otimes_R S)^{\oplus m}$ . Now, composing this embedding with  $\rho_U$ , we obtain a closed embedding  $G \hookrightarrow R_{V/U}(GL_{m,V}) \hookrightarrow GL_{n,U}$ , for  $n = md$ .

□

**Theorem 3.2.** *Let  $B$  be a semi-local Noetherian ring containing an infinite field. Let  $G$  be an isotropic simply connected simple group scheme over  $B$ . Let  $P$  be a principal  $G$ -bundle over  $\mathbb{A}_B^1$  trivial over  $(\mathbb{A}_B^1)_f$  for a monic polynomial  $f \in B[t]$ . Then  $P$  is trivial.*

*Proof.* This theorem was proved in [PaStV]. Indeed, this is precisely [PaStV, Theorem 2.1], except that in that theorem the base ring  $B$  was required to be “of geometric type”, i.e. a semilocal ring of finitely many points on a smooth variety over an infinite field. However, tracing the proof of this statement, one readily sees that the only properties of  $B$  that are used are that  $B$  is semi-local, Noetherian, and contains an infinite field. (The “geometric type” assumption was an umbrella assumption in the most part of [PaStV], since it is crucial for the validity of the main theorem [PaStV, Theorem 1.2].)  $\square$

*Proof of Theorem 1.2.* Consider the case where  $R$  is a semi-local ring of several points on a  $k$ -smooth scheme over an infinite field  $k$  (the “geometric case”). Let  $P$  be a principal  $G$ -bundle which is in the kernel of the map

$$H_{\text{ét}}^1(\mathcal{A} \times_{\text{Spec } k} \text{Spec } R, G) \rightarrow H_{\text{ét}}^1(\mathcal{A} \times_{\text{Spec } k} \text{Spec } K, G).$$

By considerations in § 2 there is a principal  $G$ -bundle  $P_t$  over  $\mathcal{A} \times_k \mathbb{A}_R^1 = \mathbb{A}_{\mathcal{A} \otimes_k R}^1$ , trivial over  $\mathcal{A} \times_k (\mathbb{A}_R^1)_f$  for a monic polynomial  $f \in R[t]$ , and such that if  $P_t$  is trivial on the whole  $\mathcal{A} \times_k \mathbb{A}_R^1$ , then the original  $G$ -bundle  $P$  over  $\mathcal{A} \times_k \text{Spec } R$  is trivial as well. Thus, it is enough to show that  $P_t$  is trivial.

Since  $R$  is a semilocal ring containing an infinite field, and  $f$  is monic, the Chinese remainder theorem implies that there is  $a \in R$  with  $f(a) \in R^\times$ ; changing the variable, we can assume that  $f(0) \in R^\times$ .

Set  $B = \mathcal{A} \otimes_k R$ . Note that  $B$  is a Noetherian commutative ring containing an infinite field  $k$ . By Theorem 3.2 for any localization  $B_m$  of  $B$  at a maximal ideal  $m \subseteq B$ , the bundle  $P_t \times_{\text{Spec } B} \text{Spec } B_m$  is trivial. By Lemma 3.1 the group scheme  $G$  admits a closed embedding into some  $\text{GL}_n$  over  $R$ , and hence, by base change, over  $B$ . Thus, we are given a principal  $G$ -bundle  $P_t$  over  $\mathbb{A}_B^1 = \mathbb{A}_k^1 \times_k \text{Spec } B$ , which is trivial Zariski-locally in  $\text{Spec } B$ , as well as on  $\{0\} \times \text{Spec } B$ ; and  $G$  is a linear group. Then by Moser’s local-global principle [Mo, Korollar 3.5.2]  $P_t$  is trivial on  $\mathbb{A}_B^1 = \mathcal{A} \times_k \mathbb{A}_R^1$ .  $\square$

*Proof of Theorem 1.1.* The claim follows from Theorem 1.2 via the well-known result of D. Popescu [Po, Sw]. Since the field  $k$  is perfect, the morphism  $k \rightarrow R$  is geometrically regular. Therefore, by Popescu’s theorem  $R$  is a filtered direct limit of smooth  $k$ -algebras. One readily sees that, since  $R$  is semilocal, these smooth  $k$ -algebras can also be chosen to be semilocal rings of several points on a smooth  $k$ -variety. Since the functor  $H_{\text{ét}}^1(-, G)$  commutes with filtered direct limits, the result follows.  $\square$

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